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Revisiting several problems and algorithms in continuous location with ℓ_τ norms

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Abstract This paper addresses the general continuous single facility location problems in finite dimension spaces under possibly different ℓ_τ norms, $\tau \geq 1$, in the demand points. We analyze the difficulty of this family of problems and revisit convergence properties of some well-known algorithms. The ultimate goal is to provide a common approach to solve the family of continuous ℓ_τ ordered median location problems Nickel and Puerto (Facility location: a unified approach, 2005) in dimension d (including of course the ℓ_τ minisum or Fermat-Weber location problem for any $\tau \geq 1$). We prove that this approach has a polynomial worst case complexity for monotone lambda weights and can be also applied to constrained and even non-convex problems.

Keywords Continuous location · Ordered median problems · Semidefinite programming · Moment problem

Mathematics Subject Classification 90B85 · 90C22 · 65K05 · 12Y05 · 46N10

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1 Introduction

Location analysis is a very active topic within the operations research community. It has giving rise to a number of nowadays standard optimization problems some of them in the core of modern mathematical programming. One of its branches is continuous location, a family of models directly related to important areas of mathematics such as linear and non-linear programming, convex analysis and global optimization (see e.g. [20] and the references therein). It is widely agreed that modern continuous location started with the paper by Weber [54] who first considers the minimization of weighted sums of distances as an economical goal to locate industries. This problem is currently known as Fermat–Weber, also because of the three points Fermat problem (s. XVII) firstly solved by Torricelli in 1659. The algorithmic part of this history starts at 1937 with the paper by Weiszfeld [55] who proposed an iterative gradient type algorithm to find or to approximate the solutions of the above mentioned Fermat-Weber problem.

For several decades this algorithm remains forgotten but in 1973 Kuhn [29] rediscovered it and proved its convergence, under some conditions, in the Euclidean case. 1 year later Katz [30] gives another convergence result. Several years later, a number of authors considered the weighted minisum problem under different norms mainly ℓ_τ or polyhedral (see e.g. [20] for a detailed literature review) and Chandrasekaran and Tamir [12] raise several interesting questions concerning resolubility of Weiszfeld algorithm. Eventually, starting in the nineties, several authors were very interested in proving the convergence of some modifications of the Weiszfeld algorithm, usually called modified Weiszfeld or generalized iterative procedure for minisum location problems.

The convergence for Euclidean distances ($\tau = 2$) was studied later by [29,30], among others. Since then, we can find in the literature many references concerning this algorithm, as for instance the generalization to ℓ_τ distances with $\tau \in [1, 2]$ [39] or the analysis of its local and global convergence [4–6,10]. Also, these results were extended to more general problems: on Banach spaces [23,45,46], on the sphere [56], with regional demand [13,52], with sets as demand facilities and using closest Euclidean distances [7] or with radial distances [14–16,31,40]. In addition, one can find in the literature papers where the convergence is accelerated using alternative step sizes [17,18,25,30,43] or some related properties concerning the termination of the algorithm in any of the demand points after a finite number of iterations [6,8,9,11,12, 29].

The influence of Weiszfeld algorithm in Location Analysis has been rather important very likely due to its very easy implementation. For several years, it was a very effective method to solve minisum continuous location problems, even though its theoretical convergence was not proven. Thus, locators have devoted a lot of effort to prove its convergence. The global convergence result of this algorithm for ℓ_τ, $\tau \in [1, 2]$, was proved in [5] and recently [50] has given a proof to close the cases ($\tau > 2$) that were not yet justified. This has been an important effort from a mathematical point of view. Nevertheless, pursuing this goal locators did not focus on the origin of the problem, namely to search for alternative, efficient algorithms to solve the ℓ_τ minisum and some more general families of location problems.

The situation is even harder if we consider a more general family of location problems that have attracted a lot of attention in the field in the last years, namely continuous ordered median location problems [42]. Ordered median problems represent as special cases nearly all classical objective functions in location theory, including the Median, CentDian, center and k-centra. More precisely, the 1-facility ordered median problem can be formulated as follows: A vector of weights $(\lambda_1, \ldots, \lambda_n)$ is given. The problem is to find a location for a facility that minimizes the weighted sum of distances where the distance to the closest point to the facility is multiplied by the weight λ_n, the distance to the second closest, by λ_{n-1}, and so on. The distance to the farthest point is multiplied by λ_1. Many location problems can be formulated as the ordered 1-median problem by selecting appropriate weights. For example, the vector for which all $\lambda_i = 1$ is the unweighted 1-median problem, the problem where $\lambda_1 = 1$ and all others are equal to zero is the 1-center problem, the problem where $\lambda_1 = \ldots = \lambda_k = 1$ and all others are equal to zero is the k-centrum. Minimizing the range of distances is achieved by $\lambda_1 = 1,\ \lambda_n = -1$ and all others are zero. Despite its full generality, the main drawback of this framework is the difficulty of solving the problems with a unified tool. There have been some successful approaches that are now available whenever the framework space is either discrete (see [3,37]) or a network (see [27,28,41] or [47]). Nevertheless, the continuous case has been, so far, only partially covered even under the additional hypothesis of convexity. There have been some attempts to overcome this drawback and there are nowadays some available methodologies to tackle these problems, at least in the plane and with Euclidean norm. In Drezner [19] and Drezner and Nickel [21,22] the authors present two different approaches. The first one uses a geometric branch and bound method based on triangulations (BTST) and the second one on a D-C decomposition for the objective function that allow solving problems on the plane. Espejo et al. [24] also address the unconstrained convex ordered median location problem on the plane and Rodriguez-Chia et al. [49] attacks the k-centrum problem using geometric arguments and developing a better algorithm applicable only for that unconstrained problem on the plane and Euclidean distances. More recently, Blanco et al. [2] have presented a new methodology based on a hierarchy of SDP relaxations that can be used to solve (approximate) the optimal solutions of the general ordered median location problems which main drawback is the size of the SDP objects that have to be used to get good accuracy in high dimension.

The above discussion points out that there exists a lack of a unified resolution approach to those problems as well as effective algorithms for the general cases. Our goal in this paper is to design a common approach to solve the family of continuous ℓ_τ, for $\tau \geq 1$, ordered median location problems in dimension d (including of course the ℓ_τ minisum or Fermat-Weber location problem for any $\tau \geq 1$). We prove that this approach has a polynomial worst case complexity for monotone lambda weights and can be also applied to constrained problems and to approximate even non-convex problems. Thus, providing a unifying new algorithmic paradigm for this class of location problems. First, for convex location problems it avoids the drawback of limit convergence proven for the Weiszfeld type algorithms. Then, it can be applied to any convex ordered median problem, even with mixed norms, in any dimension and with rather general convex constraints. Moreover, we show an explicit reformulation of these problems as SDP problems which enables the usage of standard free source solvers (SeDuMi, SDPT3,...) to solve them up to any degree of accuracy. Finally, we also show how to adapt this approach to approximate up to any degree of accuracy non-convex constrained location problems using a hierarchy of convergent relaxed problems following the rationale of [2].

The paper is organized in 5 sections. In Sect. 2 we provide a compact representation, valid for any unconstrained convex ordered location problem, by means of a new formulation that reduces these problems to semidefinite problems. This approach allows us to ensure that all these problems are polynomially solvable in finite dimension. We also present a new linear programming formulation for this problem for the norm ℓ_1. Section 3 is devoted to extend the results of Sect. 2 to the case of constrained problems under the condition of SDP-representability. Then, we handle the general case of non-convex constrained ordered median location problem for which we construct a hierarchy of SDP relaxations that converges to the optimal solution of the original problem. Our Sect. 4 is devoted to the computational experiments. We report results in four different problem types, namely minisum (Weber), minimax (center), k-centrum (minimizing the sum of the k-largest distances) and general ordered median problems. In this section we also compare our results with those obtained for the cases that have been previously reported in the literature. The paper ends, in Sect. 5, with some conclusions and an outlook for further research.

2 A compact representation of the convex ordered median problem

In this section we present the convex ordered median problem in dimension d where the distances are measured with a general ℓ_τ-norm being $\tau \geq 1$ and $\tau \in \mathbb{Q}$. Recall that the ℓ_τ-norm of a vector $x \in \mathbb{R}^d$ is defined as $\|x\|_\tau = (\sum_{j=1}^d |x_j|^\tau)^{1/\tau}$. We are given a set of demand points $S = \{a_1, ..., a_n\}$ and two sets of scalars $\Omega := \{\omega_1, ..., \omega_n\}$, $\omega_i \geq 0,\ \forall\ i \in \{1, \ldots, n\}$ and $\Lambda := \{\lambda_1, ..., \lambda_n\}$ where $\lambda_1 \geq ... \geq \lambda_n \geq 0$. The elements ω_i are weights corresponding to the importance given to the existing facilities a_i, $i \in \{1, ..., n\}$ and depending on the choice of the elements of Λ we get different classes of problems. We denote by $\mathcal{P}_n$ the set of permutations of the first n natural numbers.

Given a permutation $\sigma \in \mathcal{P}_n$ satisfying

$$\omega_{\sigma(1)}\|x - a_{\sigma(1)}\|_\tau \geq \ldots \geq \omega_{\sigma(n)}\|x - a_{\sigma(n)}\|_\tau,$$

the unconstrained ordered median problem (see [42]) consists of

$$\min_{x\in\mathbb{R}^d} \sum_{i=1}^n \lambda_i \omega_{\sigma(i)}\|x - a_{\sigma(i)}\|_\tau. \tag{1}$$

The reader may note that the representation of the above problem depends on the permutation σ that sorts the distances from the demand points a_i to the solution point x. This implies that it is not linear over the entire space $\mathbb{R}^d$ since each time that the ordering of the elements of the vector of distances, $(\omega_i\|x - a_i\|_\tau)_{i=1}^n$, changes so does the allocation of these elements to the lambda coefficients, and therefore the representation of the function. In particular, this fact makes the objective function to be non-differentiable over the boundary of ordered regions (see [42,44]). An important consequence of the above discussion is that one cannot apply unconstrained optimization (gradient type) methods to solve Problem (1).

Moreover, if the lambda weights are not sorted in non-increasing sequence the problem is not convex and therefore, it becomes hard since it includes, among others, general instances of concave minimization, which are well-known to be NP-hard. Nevertheless, even in the case of non-increasing monotone lambda the problem is not easy due to its lack of easy representability. Witness of its difficulty is the fact that so far, it has been solved, in the specialized literature of Location Analysis, only in the plane [21,24].

In order to address the resolution of the problem that we present in this paper we need to restate the problem in a lifted space of variables that allows a unified representation as a constrained optimization problem. We start by showing a compact reformulation of the above problem that will be later useful in our approach.

Theorem 1 *Let $\tau = \frac{r}{s} \geq 1$ be such that $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$ and $\gcd(r, s) = 1$. For any set of lambda weights satisfying $\lambda_1 \geq ... \geq \lambda_n$, Problem (1) is equivalent to*

$$\min \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i \tag{2}$$

$$s.t\ \ v_i + w_k \geq \lambda_k z_i, \quad \forall i, k = 1, ..., n, \tag{3}$$

$$y_{ij} - x_j + a_{ij} \geq 0, \quad i = 1, \ldots, n, \ \ j = 1, ..., d, \tag{4}$$

$$y_{ij} + x_j - a_{ij} \geq 0, \quad i = 1, \ldots, n, \ \ j = 1, ..., d, \tag{5}$$

$$y_{ij}^r \leq u_{ij}^s z_i^{r-s}, \quad i = 1, \ldots, n, \ \ j = 1, ..., d, , \tag{6}$$

$$\omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \leq z_i, \quad i = 1, \ldots, n, \tag{7}$$

$$u_{ij} \geq 0, \quad i = 1, \ldots, n, \ \ j = 1, \ldots, d. \tag{8}$$

Proof Because of the condition $\lambda_1 \geq ... \geq \lambda_n$, Problem (1) can be equivalently written as

$$\min_{x \in \mathbb{R}^d} \max_{\sigma \in \mathcal{P}_n} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau, \tag{9}$$

Let us introduce auxiliary variables z_i, $i = 1, \ldots, n$ to which we impose that $z_i \geq \omega_i \|x - a_i\|_\tau$, to model the problem in a convenient form. Now, for any permutation $\sigma \in \mathcal{P}_n$, let $z_\sigma = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. Moreover, let us denote by $(\cdot)$ the permutation that sorts any vector in nonincreasing sequence, i.e. $z_{(1)} \geq z_{(2)} \geq \ldots \geq z_{(n)}$. Using that $\lambda_1 \geq ... \geq \lambda_n$ and since $z_i \geq 0$, for all $i = 1, \ldots, n$ then

$$\sum_{i=1}^{n} \lambda_i z_{(i)} = \max_{\sigma \in \mathcal{P}_n} \sum_{i=1}^{n} \lambda_i z_{\sigma(i)}.$$

The permutations in $\mathcal{P}_n$ can be represented by the following set of equations with binary variables p_{ik}:

$$\begin{cases} \sum_{i=1}^{n} p_{ik} = 1, \ \forall k = 1, ..., n, \\ \sum_{k=1}^{n} p_{ik} = 1, \ \forall i = 1, ..., n, \end{cases} \quad \text{where} \quad p_{ik} = \begin{cases} 1, & \text{if } z_i \text{ goes in position } k, \\ 0, & \text{otherwise.} \end{cases}$$

Next, combining the two sets of variables we obtain that the objective function of (9) can be equivalently written as::

$$\begin{cases} \sum_{i=1}^{n} \lambda_i z_{(i)} = \max \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_k z_i p_{ik} \\ \qquad s.t \quad \sum_{i=1}^{n} p_{ik} = 1, \ \forall k = 1, ..., n, \\ \qquad\qquad \sum_{k=1}^{n} p_{ik} = 1, \ \forall i = 1, ..., n, \\ \qquad\qquad p_{ik} \in \{0, 1\}. \end{cases} \tag{10}$$

Now, we point out that for fixed $z_1, ..., z_n$, the above problem is an assignment problem and its constraint matrix is totally unimodular, so that solving a continuous relaxation of the problem always yields an integral solution vector (see [1]), and thus a valid permutation. Moreover, the dual of the linear programming relaxation of (10) is strong and also gives the value of the original binary formulation of (10).

Hence, for any vector $z \in \mathbb{R}^n$, by using the dual of the assignment problem (10) we obtain the following equivalent expression for (9)

$$\begin{cases} \min \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i \\ s.t \quad v_i + w_k \geq \lambda_k z_i, \qquad \forall i, k = 1, ..., n, \\ \qquad z_i \geq \omega_i \|x - a_i\|_\tau, \quad i = 1, ..., n. \end{cases} \tag{11}$$

It remains to prove that each inequality $z_i \geq \omega_i \|x - a_i\|_\tau, \ i = 1, ..., n$ can be replaced by the system:

$$\begin{aligned} y_{ij} - x_j + a_{ij} &\geq 0, \quad j = 1, ..., d. \\ y_{ij} + x_j - a_{ij} &\geq 0, \quad j = 1, ..., d. \\ y_{ij}^r &\leq u_{ij}^s z_i^{r-s}, \quad j = 1, ..., d. \\ \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} &\leq z_i, \\ u_{ij} &\geq 0, \ \forall \ j = 1, \ldots, d. \end{aligned}$$

Indeed, set $\rho = \frac{r}{r-s}$, then $\frac{1}{\rho} + \frac{s}{r} = 1$. Let $(\bar{x}, \bar{z}_i)$ fulfill the inequality $z_i \geq \omega_i \|x - a_i\|_\tau$. Then we have

$$\omega_i \|\bar{x} - a_i\|_\tau \leq \bar{z}_i \iff \omega_i \left(\sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{\frac{r}{s}} \right)^{\frac{s}{r}} \leq \bar{z}_i^{\frac{s}{r}} \bar{z}_i^{\frac{1}{\rho}}$$

$$\iff \omega_i \left(\sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{\frac{r}{s}} \bar{z}_i^{\frac{r}{s}(-\frac{r-s}{r})} \right)^{\frac{s}{r}} \leq \bar{z}_i^{\frac{s}{r}}$$

$$\iff \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{\frac{r}{s}} \bar{z}_i^{-\frac{r-s}{s}} \leq \bar{z}_i \tag{12}$$

Then (12) holds if and only if $\exists u_i \in \mathbb{R}^d,\ u_{ij} \geq 0,\ \forall j = 1, ..., d$ such that

$$|\bar{x}_j - a_{ij}|^{\frac{r}{s}} \bar{z}_i^{-\frac{r-s}{s}} \leq u_{ij}, \quad \text{satisfying} \quad \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \leq \bar{z}_i,$$

or equivalently,

$$|\bar{x}_j - a_{ij}|^r \leq u_{ij}^s \bar{z}_i^{r-s}, \quad \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \leq \bar{z}_i. \tag{13}$$

Set $\bar{y}_{ij} = |\bar{x}_j - a_{ij}|$ and $\bar{u}_{ij} = |\bar{x}_j - a_{ij}|^\tau \bar{z}_i^{-1/\rho}$. Then, clearly $(\bar{x}, \bar{z}_i, \bar{y}, \bar{u})$ satisfies (4)–(8).

Conversely, let $(\bar{x}, \bar{z}_i, \bar{y}, \bar{u})$ be a feasible solution of (4)–(8). Then, $\bar{y}_{ij} \geq |\bar{x}_{ij} - a_{ij}|$ for all i, j and by (6) $\bar{u}_{ij} \geq \bar{y}_{ij}^{(r/s)} z_i^{-\frac{r-s}{s}} \geq |\bar{x}_j - a_{ij}|^\tau \bar{z}_i^{-\frac{r-s}{s}}$. Thus,

$$\omega_i^{r/s} \sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{r/s} \bar{z}_i^{-\frac{r-s}{s}} \leq \omega_i^{r/s} \sum_{j=1}^{d} \bar{u}_{ij} \leq \bar{z}_i,$$

which in turns implies that $\omega_i^{r/s} \sum_{j=1}^{d} |\bar{x}_j - a_{ij}|^{r/s} \leq \bar{z}_i \bar{z}_i^{\frac{r-s}{s}}$ and hence, $\omega_i \|\bar{x} - a_i\| \leq \bar{z}_i$. □

Problem (2)–(8) is an exact representation of Problem (1) in any dimension and for any ℓ_τ-norm, $\tau \geq 1$.

For the case $\tau = 1$, the above problem reduces to a linear programming problem. We observe that even though for the case $\tau = 1$, it was known that these problems reduce to linear programming [42], the formulation below is new and different from those presented in the literature [28,42].

Corollary 1 *If $\tau = 1$ the reformulation given by Problem (1) is*

$$\begin{cases} \min \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i \\ s.t \quad v_i + w_k \geq \lambda_k z_i, \ \forall i, k = 1, ..., n, \\ \quad z_i \geq \omega_i \sum_{j=1}^{d} u_{ij}, \quad i = 1, ..., n, \\ \quad x_j - a_{ij} \leq u_{ij} \quad i = 1, ..., n, \ j = 1, \dots, d, \\ \quad -x_j + a_{ij} \leq u_{ij} \quad i = 1, ..., n, \ j = 1, \dots, d. \end{cases} \tag{14}$$

The reader may observe that the representation given in Theorem 1 is new and different from the one used in [2]. On the one hand, this new formulation is more efficient than the one presented in [2] and specially tailored for the case of non-increasing monotone lambda weights, see e.g. [2, Lemma 8]. For the sake of readability we include it in the following.

Let

$$S_k(x) := \sum_{j=1}^{k} z_{(j)}, \tag{15}$$

where $z_{(j)}$ is such that $z_{(1)} \geq \dots \geq z_{(n)}$. That formulation applied to the setting of this paper reads as:

$$\begin{aligned} \min \quad & \textstyle\sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) S_k(x) \\ & t_k + r_{kj} \geq z_j(x), \quad j, k = 1, \dots, n, \\ & r_{kj} \geq 0, \quad j, k = 1, \dots, n, \\ & z_j \geq \omega_j \|x - a_j\|_\tau, \quad j = 1, ..., n. \end{aligned} \tag{16}$$

It is easy to see that formulation (16) has $O(n^2+d)$ variables and $O(n^2)$ constraints whereas the new one written in similar terms as presented in (11) has $O(n+d)$ variables and $O(n^2)$ constraints.

Our goal is to show that for any $\tau > 1$, $\tau \in \mathbb{Q}$, Problem (2)–(8) also admits a compact formulation within another easy class of polynomially solvable mathematical programming problems: Semidefinite Programming Problems. In order to get that we need to prove a technical lemma. Let $\#A$ denote the cardinality of the set A.

Lemma 1 *Let $\tau = \frac{r}{s} > 1$, $\tau \neq 2$ be such that $r, s \in \mathbb{N} \setminus \{0\}$ and $\gcd(r, s) = 1$. Let x, u and t be non negative and satisfying*

$$x^r \leq u^s t^{r-s}. \tag{17}$$

Assume that $2^{k-1} < r \leq 2^k$ where $k \in \mathbb{N} \setminus \{0\}$ such that

$$x^{2^k} \leq u^s t^{r-s} x^{2^k - r}, \tag{18}$$

and

$$s = \alpha_{k-1}2^{k-1} + \alpha_{k-2}2^{k-2} + \ldots + \alpha_1 2^1 + \alpha_0 2^0, \tag{19}$$
$$r - s = \beta_{k-1}2^{k-1} + \beta_{k-2}2^{k-2} + \ldots + \beta_1 2^1 + \beta_0 2^0, \tag{20}$$
$$2^k - r = \gamma_{k-1}2^{k-1} + \gamma_{k-2}2^{k-2} + \ldots + \gamma_1 2^1 + \gamma_0 2^0, \tag{21}$$

where $\alpha_i,\ \beta_i,\ \gamma_i \in \{0, 1\}$.

Then, if (x, t, u) *is a feasible solution of (17) there exists* w *such that either*

1. (x, t, u, w) *is a solution of System (22), if* $\alpha_i + \beta_i + \gamma_i = 1$, *for all* $0 < i \le k-1$.

$$\begin{cases} w_1^2 & \le u^{\alpha_0} t^{\beta_0} x^{\gamma_0}, \\ w_{i+1}^2 & \le w_i u^{\alpha_i} t^{\beta_i} x^{\gamma_i}, \quad i = 1, \ldots, k-2 \\ x^2 & \le w_{k-1} u^{\alpha_{k-1}} t^{\beta_{k-1}} x^{\gamma_{k-1}}, \end{cases} \tag{22}$$

2. *Let* $c = \#\{i : \alpha_i + \beta_i + \gamma_i = 3,\ i = 2, ..., k-2\}$, (x, t, u, w) *is a solution of System (23), if there exist* i_j *and* $i_{l(j)},\ j = 1, \ldots, c$ *such that:*
   1. $0 < i_1 < i_2 < \ldots < i_c \le k-2$,
   2. $i_j < i_{l(j)} < i_{j+1}$,
   3. $\alpha_{i_j} + \beta_{i_j} + \gamma_{i_j} = 3$, $\alpha_{i_{l(j)}} + \beta_{i_{l(j)}} + \gamma_{i_{l(j)}} = 0$ *and* $\alpha_h + \beta_h + \gamma_h = 2$ *for* $h = i_j + 1, \ldots, i_{l(j)-1}$.

$$\begin{cases} w_1^2 & \le u^{\alpha_0} t^{\beta_0} x^{\gamma_0}, \\ w_{i+1}^2 & \le w_i u^{\alpha_i} t^{\beta_i} x^{\gamma_i},\ i \in \{1, \ldots, i_1 - 1\} \\ \text{-------- } \textit{for each } j = 1, \ldots, c \text{ --------} & \\ w_{\theta(j)}^2 & \le ut, \\ w_{\theta(j)+1}^2 & \le w_{\theta(j)-1} x \\ \left.\begin{aligned} w_{\theta(j)+2*s}^2 & \le w_{\theta(j)+2(s-1)} a_{i_j+s} \\ w_{\theta(j)+2*s+1}^2 & \le w_{\theta(j)+2s-1} b_{i_j+s} \end{aligned}\right\}, & \begin{aligned} & s = 1, \ldots, i_{l(j)} - i_j - 1 \textit{ and} \\ & a_{i_j+s} + b_{i_j+s} = u^{\alpha_{i_j+s}} t^{\beta_{i_j+s}} x^{\gamma_{i_j+s}}, \end{aligned} \\ w_{\theta(j)+2(i_{l(j)}-i_j)}^2 & \le w_{\theta(j)+2(i_{l(j)}-i_j-1)} w_{\theta(j)+2(i_{l(j)}-i_j-1)+1}, \begin{cases} \textit{if } m-1 \ge \\ \theta(j) + 2(i_{l(j)} - i_j - 1) \end{cases}, \\ w_{\theta(j)+2(i_{l(j)}-i_j)+s}^2 & \le w_{\theta(j)+2(i_{l(j)}-i_j)+s-1} u^{\alpha_{i_{l(j)}+s}} t^{\beta_{i_{l(j)}+s}} x^{\gamma_{i_{l(j)}+s}}, \begin{cases} \textit{for all } s = 1, \ldots, \\ i_{j+1} - i_{l(j)} - 1 \end{cases} \\ \text{--------------------------------} & \\ x^2 & \le w_m d. \end{cases} \tag{23}$$

where $\theta = (\theta(j))_{j=1}^c$ *such that* $\theta(j) = 2\#\{i : \alpha_i + \beta_i + \gamma_i \ge 2,\ 1 \le i \le i_j\} + \#\{i : \alpha_i + \beta_i + \gamma_i \le 1,\ 1 \le i \le i_j\}$ *for* $j = 1, ..., c$, $m = 1 + 2\#\{i : \alpha_i + \beta_i + \gamma_i \ge 2, 1 \le i < k-1\} + \#\{i : \alpha_i + \beta_i + \gamma_i \le 1, 1 \le i < k-1\} \le 2k$ *and* $d = \begin{cases} w_{m-1} & \textit{if } \alpha_{k-1} + \beta_{k-1} + \gamma_{k-1} = 0 \\ u^{\alpha_{k-1}} t^{\beta_{k-1}} x^{\gamma_{k-1}} & \textit{if } \alpha_{k-1} + \beta_{k-1} + \gamma_{k-1} = 1 \end{cases}$. *Conversely, if* (x, t, u, w) *is a solution of (22) or (23) then* (x, t, u) *is a feasible solution of (17).*

Proof To get the expressions of any of the systems (22) or (23), we discuss the decomposition (19), (20), (21) of s, $r - s$ and $2^k - r$ in the basis $B = \{2^l\}$, $l = 0, ..., k-1$.

Since $2^k = 2^k - r + (r - s) + s$, we observe that (19)+(20)+(21) gives a decomposition of 2^k in power of 2 summands with coefficients less than or equal than 3. Namely,

$$2^k = (\alpha_{k-1} + \beta_{k-1} + \gamma_{k-1})2^{k-1} + \ldots + (\alpha_1 + \beta_1 + \gamma_1)2^1 + (\alpha_0 + \beta_0 + \gamma_0)2^0. \tag{24}$$

We discuss two cases depending on the parity of r.

If r is even then s is odd since $\gcd(r, s) = 1$, thus $r - s$ is odd and $2^k - r$ is even.

If r is odd then s can be odd or even; if s is odd then $r - s$ is even and $2^k - r$ is odd; otherwise $r - s$ is odd and $2^k - r$ is odd.

From the above discussion, we observe that there are always two odd and one even numbers in the triplet $(s, r - s, 2^k - r)$. Therefore, we conclude that

$$\alpha_0 + \beta_0 + \gamma_0 = 2.$$

On the other hand, since $\sum_{i=0}^{k-1} 2^i = 2^k - 1$, then another representation of 2^k is:

$$2^k = 1.2^{k-1} + 1.2^{k-2} + \ldots + 1.2^1 + 2.2^0. \tag{25}$$

Considering the fact that (25) and (24) are two representations of 2^k, by equating coefficients, we deduce some properties of the sums $(\alpha_i + \beta_i + \gamma_i)$, $i = 1, ..., k - 1$.

- First of all, we observe that $\alpha_{k-1} + \beta_{k-1} + \gamma_{k-1}$ can only assume the values 0 or 1.
- Second, since $\alpha_0+\beta_0+\gamma_0 = 2$ then it implies that $\alpha_1+\beta_1+\gamma_1 = 1$ *or* 3, otherwise if $\alpha_1 + \beta_1 + \gamma_1 = 0$ *or* 2, then we will get $0.2^1 = 0$ or $2.2^1 = 2^2$ which means that we can not recover the term 2^1 and then we will not get the decomposition as in (25).
- Third, let i_0 be the first index, counting in a decreasing order from $k - 1$ to 1, so that $\alpha_{i_0} + \beta_{i_0} + \gamma_{i_0} \neq 1$ and $\forall\, i,\ i_0 < i \leq k - 1$ we have $\alpha_i + \beta_i + \gamma_i = 1$. Then three cases can occur:
  1. if $\alpha_{i_0} + \beta_{i_0} + \gamma_{i_0} = 3$, then

$$\begin{aligned} 2^k &= 1.2^{k-1} + ... + 1.2^{i_0+1} + 3.2^{i_0} + (\alpha_{i_0-1}+\beta_{i_0-1}+\gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0,\\ &= 1.2^{k-1} + ... + 2.2^{i_0+1} + 1.2^{i_0} + (\alpha_{i_0-1}+\beta_{i_0-1}+\gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0,\\ &\vdots\\ &= 2.2^{k-1} + ... + 0.2^{i_0+1} + 1.2^{i_0} + (\alpha_{i_0-1}+\beta_{i_0-1}+\gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0, \end{aligned}$$

  which it is not possible and therefore it implies that $\alpha_{i_0} + \beta_{i_0} + \gamma_{i_0} = 2$ *or* 0.
  2. if $\alpha_{i_0} + \beta_{i_0} + \gamma_{i_0} = 2$, then

$$\begin{aligned} 2^k &= 1.2^{k-1} + ... + 1.2^{i_0+1} + 2.2^{i_0} + (\alpha_{i_0-1}+\beta_{i_0-1}+\gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0,\\ &= 1.2^{k-1} + ... + 2.2^{i_0+1} + 0.2^{i_0} + (\alpha_{i_0-1}+\beta_{i_0-1}+\gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0,\\ &\vdots\\ &= 2.2^{k-1} + ... + 0.2^{i_0+1} + 0.2^{i_0} + (\alpha_{i_0-1}+\beta_{i_0-1}+\gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0, \end{aligned}$$

which again it is not possible and therefore it implies that $\alpha_{i_0} + \beta_{i_0} + \gamma_{i_0} = 0$. From the above two cases, we summarize that the first sum $\alpha_{i_0} + \beta_{i_0} + \gamma_{i_0} \neq 1$ must be necessarily $\alpha_{i_0} + \beta_{i_0} + \gamma_{i_0} = 0$. Based on this we consider the only possible case.

3. if $\alpha_{i_0}+\beta_{i_0}+\gamma_{i_0} = 0$, then it must exist $i_1 < i_0$, satisfying that $\alpha_{i_1}+\beta_{i_1}+\gamma_{i_1} = 3$ and such that for all $k,\ i_1 < k \leq i_0,\ \alpha_k + \beta_k + \gamma_k = 2$. Indeed,
   (a) if $\alpha_{i_0-1} + \beta_{i_0-1} + \gamma_{i_0-1} = 1$, then

$$\begin{aligned}2^k &= 1.2^{k-1} + ... + 1.2^{i_0+1} + 0.2^{i_0} + (\alpha_{i_0-1} + \beta_{i_0-1} + \gamma_{i_0-1})2^{i-1} + ... + 2.2^0,\\ &= 1.2^{k-1} + ... + 1.2^{i_0+1} + 0.2^{i_0} + 1.2^{i_0-1} + ... + 2.2^0.\end{aligned}$$

   Hence, since the sums $\alpha_j + \beta_j + \gamma_j \leq 3$ for all j, one cannot recover the sum 2^k in (25) and the representation of 2^k would be wrong.

   (b) if $\alpha_{i_0-1} + \beta_{i_0-1} + \gamma_{i_0-1} = 3$, then

$$\begin{aligned}2^k &= 1.2^{k-1} + ... + 1.2^{i_0+1} + 0.2^{i_0} + (\alpha_{i_0-1} + \beta_{i_0-1} + \gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0,\\ &= 1.2^{k-1} + ... + 0.2^{i_0} + 3.2^{i_0-1} + ... + 2.2^0,\\ &= 1.2^{k-1} + ... + 1.2^{i_0} + 1.2^{i_0-1} + ... + 2.2^0.\end{aligned}$$

   The representation of 2^k would be valid until the term $i_0 - 1$ and we can repeat the argument with the next element whose coefficient is different in the representation of 2^k in (24) and (25).

   (c) if $\alpha_{i_0-1} + \beta_{i_0-1} + \gamma_{i_0-1} = 2$, then

$$\begin{aligned}2^k &= 1.2^{k-1} + ... + 1.2^{i_0+1} + 0.2^{i_0} + (\alpha_{i_0-1} + \beta_{i_0-1} + \gamma_{i_0-1})2^{i_0-1} + ... + 2.2^0,\\ &= 1.2^{k-1} + ... + 0.2^{i_0} + 2.2^{i_0-1} + ... + 2.2^0,\\ &= 1.2^{k-1} + ... + 1.2^{i_0} + 0.2^{i_0-1} + ... + 2.2^0,\end{aligned}$$

   This way we get that the representations of 2^k are equal until the term i_0. Next, to recover the term 2^{i_0-1} then $\alpha_{i_0-2} + \beta_{i_0-2} + \gamma_{i_0-2} = 2$ or 3 so that we are in cases (b) or (c) and we repeat until we get the decomposition (25).

The analysis above justifies that the only possible cases in any representation of 2^k in the form $(2^k - r) + (r - s) + s$ and each of the addends ($2^k - r$, $r - s$ and s) in the basis $B = \{2^l\}$, $l = 0, \ldots, k-1$ are those that correspond to cases 1 or 2 in the statement of the lemma.

Let m denote the number of inequalities in any of the systems (22) or (23). First of all, we observe that the last inequality has a common form in any of the systems, namely $x^2 \leq w_m d$. Indeed, if $\alpha_{k-1} + \beta_{k-1} + \gamma_{k-1} = 0$ then we shall consider the inequality

$$x^2 \leq w_m w_{m-1} \tag{26}$$

otherwise i.e if $\alpha_{k-1} + \beta_{k-1} + \gamma_{k-1} = 1$ then we shall consider the inequality

$$x^2 \leq w_m u^{\alpha_{k-1}} t^{\beta_{k-1}} x^{\gamma_{k-1}}. \tag{27}$$

Based on the above observation, we have that the systems (22) and (23) always include (26) or (27) and other inequalities depending on the cases. Let us analyze the two cases.

Case 1 Let (x, t, u) be a solution of system (18) and $\alpha_i + \beta_i + \gamma_i = 1$ for all $0 < i < k-1$. Set $w_1 = \sqrt{u^{\alpha_0} t^{\beta_0} x^{\gamma_0}}$ and $w_{i+1} = \sqrt{w_i u^{\alpha_i} t^{\beta_i} x^{\gamma_i}}$, $i = 2, \ldots, k-2$. Clearly, (x, t, u, w) is a solution of system (22). Conversely, if (x, t, u, w) is a solution of system (22) then propagating backward from the last inequality to the first one we prove that (x, t, u) is also a feasible solution of (18).
Finally, it is clear that in this case, m, the number of inequalities necessary to represent (18) as system (22) is $m = k - 1$.
Case 2 Let (x, t, u) be a solution of system (18) and $\alpha_i + \beta_i + \gamma_i$ for all $0 < i < k - 1$ satisfying the hypotheses of Item *2.* in the thesis of the lemma. Set $w_1 = \sqrt{u^{\alpha_0} t^{\beta_0} x^{\gamma_0}}$ and w_{i+1} for $i = 2, \ldots, m$ being defined recursively according to the inequalities in (27) from the previous values of $w_j,\ j = 1, \ldots, i$, and u, t, x. Clearly, (x, t, u, w) is a solution of system (22).
Conversely, if (x, t, u, w) is a solution of system (27) then propagating backward from the last inequality to the first one we prove that (x, t, u) is also a feasible solution of (18).
We conclude the proof observing that the number of inequalities m in any of the two representations is fixed and it is equal to $m = 1 + 2\#\{i : \alpha_i + \beta_i + \gamma_i \geq 2, 1 \leq i < k - 1\} + \#\{i : \alpha_i + \beta_i + \gamma_i \leq 1, 1 \leq i < k - 1\} \leq 2k$. □

The reader should observe that the exclusion of the case $\tau = 2$ does not means any loss of generality since in this case $r = 2$, $s = 1$ and therefore the inequality (17), namely $x^2 \leq ut$ gives directly the representation without using the auxiliary w variables.

We illustrate the application of the above lemma with the following example.

Example 1 Let us consider $\tau = \frac{100000}{70001}$ which in turns means that $r = 10^5$ and $s = 70001$.

$$x^{100000} \leq u^{70001} t^{29999},$$
$$x^{2^{17}} = x^{131072} \leq u^{70001} t^{29999} x^{31072}.$$

The representations of the exponents of u, t, x in the inequality above in power of 2 summands are:

$$\begin{aligned}
u: \quad 70001 &= 1.2^{16} + 0.2^{15} + 0.2^{14} + 0.2^{13} + 1.2^{12} + 0.2^{11} + 0.2^{10} + 0.2^{9} + 1.2^{8} \\
&\quad + 0.2^{7} + 1.2^{6} + 1.2^{5} + 1.2^{4} + 0.2^{3} + 0.2^{2} + 0.2^{1} + 1.2^{0} \\
t: \quad 29999 &= 0.2^{16} + 0.2^{15} + 1.2^{14} + 1.2^{13} + 1.2^{12} + 0.2^{11} + 1.2^{10} + 0.2^{9} + 1.2^{8} \\
&\quad + 0.2^{7} + 0.2^{6} + 1.2^{5} + 0.2^{4} + 1.2^{3} + 1.2^{2} + 1.2^{1} + 1.2^{0} \\
x: \quad 31072 &= 0.2^{16} + 0.2^{15} + 1.2^{14} + 1.2^{13} + 1.2^{12} + 1.2^{11} + 0.2^{10} + 0.2^{9} + 1.2^{8} \\
&\quad + 0.2^{7} + 1.2^{6} + 1.2^{5} + 0.2^{4} + 0.2^{3} + 0.2^{2} + 0.2^{1} + 0.2^{0}
\end{aligned}$$

From the above decomposition, we realize that this example falls in case 2 and we obtain $c = 3$. The table below shows the corresponding indexes of the w-inequalities of each bloc $i_j,\ j = 1, 2, 3$.

$i_1 = 5$	$i_2 = 8$	$i_3 = 12$
$i_{l(1)} = 7$	$i_{l(2)} = 9$	$i_{l(3)} = 15$
$\theta(1) = 6$	$\theta(2) = 11$	$\theta(3) = 16$

,

the total number of inequalities is $m = 1 + 2 * 6 + 9 = 22$.

Then we get the decomposition

level 1	level 2	level 3	level 4	level 5
$w_1^2 \le ut$	$w_2^2 \le w_1 t$	$w_3^2 \le w_2 t$	$w_4^2 \le w_3 t$	$w_5^2 \le w_4 t$

Bloc i_1

level 6	level 7	level 8
$w_6^2 \le ut$	$w_8^2 \le w_6 u$	$w_{10}^2 \le w_8 w_9$
$w_7^2 \le w_5 x$	$w_9^2 \le w_7 x$	

Bloc i_2

level 8	level 9	level 10
$w_{10}^2 \le w_8 w_9$	$w_{11}^2 \le ut$	$w_{13}^2 \le w_{11} w_{12}$
	$w_{12}^2 \le w_{10} x$	

level 11	level 12
$w_{14}^2 \le w_{13} t$	$w_{15}^2 \le w_{14} x$

Bloc i_3

level 13	level 14	level 15	level 16
$w_{16}^2 \le ut$	$w_{18}^2 \le w_{16} t$	$w_{20}^2 \le w_{18} t$	$w_{22}^2 \le w_{20} w_{21}$
$w_{17}^2 \le w_{15} x$	$w_{19}^2 \le w_{17} x$	$w_{21}^2 \le w_{19} x$	

level 17
$x^2 \le w_{22} u$

From that set of inequalities one can easily obtain the original inequality by expanding backward, starting from the last level (level 17). Indeed,

level 17	level 16	level 15	level 14
$x^2 \le w_{22} u$	$x^{2^2} \le u^2 w_{20} w_{21}$	$x^{2^3} \le u^4 t x w_{18} w_{19}$	$x^{2^4} \le u^8 t^3 x^3 w_{16} w_{17}$

level 13	level 12	level 11
$x^{2^5} \le u^{17} t^7 x^7 w_{15}$	$x^{2^6} \le u^{34} t^{14} x^{15} w_{14}$	$x^{2^7} \le u^{68} t^{29} x^{30} w_{13}$

level 10	level 9	level 8
$x^{2^8} \leq u^{136} t^{58} x^{60} w_{11} w_{12}$	$x^{2^9} \leq u^{273} t^{117} x^{121} w_{10}$	$x^{2^{10}} \leq u^{546} t^{234} x^{242} w_8 w_9$

level 7	level 6	level 5
$x^{2^{11}} \leq u^{1093} t^{468} x^{485} w_6 w_7$	$x^{2^{12}} \leq u^{2187} t^{937} x^{971} w_5$	$x^{2^{13}} \leq u^{4375} t^{1874} x^{1942} w_4$

level 4	level 3	level 2
$x^{2^{14}} \leq u^{8750} t^{3749} x^{3884} w_3$	$x^{2^{15}} \leq u^{17500} t^{7499} x^{7768} w_2$	$x^{2^{16}} \leq u^{35000} t^{14999} x^{15536} w_1$

level 1
$x^{2^{17}} \leq u^{70001} t^{29999} x^{31072}$

Remark 1 The particular case of the Euclidean norm ($\tau = 2$) leads to a simpler representation based on a direct application of Schur complement.

Observe that the constraint $z_i^2 \geq \omega_i^2 \|x - a_i\|_2^2 = \omega_i^2 \sum_{j=1}^{d} (x_j - a_{ij})^2, \; i = 1, ..., n$ can be written as $L_i \succeq 0$, being

$$L_i = \begin{pmatrix} z_i - \omega_i(x_1 - a_{i1}) & \omega_i(x_2 - a_{i2}) & \cdots & \omega_i(x_d - a_{id}) \\ \omega_i(x_2 - a_{i2}) & z_i + \omega_i(x_1 - a_{i1}) & & 0 \\ \vdots & & \ddots & \\ \omega_i(x_d - a_{id}) & 0 & & z_i + \omega_i(x_1 - a_{i1}) \end{pmatrix}.$$

(Recall that for a symmetric matrix A, $A \succeq 0$ means A to be positive semidefinite.)

Next, based on Lemma 1 we can state the final representation result for the family of convex ordered continuous single facility location problems. This result is rather useful because reduces this family of problem to SDP and therefore, it will allow us to prove convergence results for solving them.

Theorem 2 *For any set of lambda weights satisfying $\lambda_1 \geq ... \geq \lambda_n$ and $\tau = \frac{r}{s}$ such that $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$ and $\gcd(r, s) = 1$, Problem (1) can be represented as a semidefinite programming problem with $n^2 + n(2d + 1)$ linear constraints and at most $4nd \log r$ positive semidefinite constraints.*

Proof Using Theorem 1 we have that Problem (1) is equivalent to Problem (2)–(8).

Then, we use Lemma 1 to represent each one of the inequalities (6) for each i, j, as a system of at most $2 \log r$ inequalities of the form (22) or (23). Next, we observe that all the inequalities that appear in (22) or (23) are of the form $a^2 \leq bc$, involving 3 variables, a, b, c with b, c non negative. Finally, it is well-known by Schur complement that

$$a^2 \leq bc \quad \Leftrightarrow \begin{pmatrix} b + c & 0 & 2a \\ 0 & b + c & b - c \\ 2a & b - c & b + c \end{pmatrix} \succeq 0, \; b + c \geq 0.$$

Hence, Problem (1) is a SDP because it has a linear objective function, $n^2 + n(2d+1)$ linear inequalities and at most $2nd\log(r)$ linear matrix inequalities. □

There is an interesting observation that follows from the above result. It was already known that continuous convex ordered location problems with ℓ_1 norm were reducible to linear programming (see e.g. [42]). This paper proves that most continuous convex ordered location problems with ℓ_p norms are reducible to SDP programming showing the similarities existing between all this class of problems and moreover that convex continuous single facility location problems are among the "easy" optimization problems. Moreover, since the inequalities that appear in Problem (2)–(8) are either linear or of the form $a^2 \le bc$, which can be equivalently written as $\left\| \begin{pmatrix} 2a \\ b-c \end{pmatrix} \right\|_2 \le b+c$, we conclude that these problems can be also cast, indeed, as second-order conic programming problems (SOCP). We shall compare the performance of our model implemented as SDP or SOCP in Sect. 4 within the computational experience.

Finally, Theorem 2 allows us to apply the general theory of conic programming to derive a general result of convergence for solving the family of continuous convex ordered single facility location problems: Problem (1) is polynomially solvable in dimension $d \in \mathbb{N}$ and for any set of nonincreasing lambda weights. Moreover, we can be more precise and can state the following result:

Theorem 3 *Let $\varepsilon > 0$ be a prespecified accuracy and (X^0, S^0) be a feasible primal-dual pair of initial solutions of Problem (2–8). An optimal primal-dual pair (X, S) satisfying $X \cdot S \le \varepsilon$ can be obtained in at most $O(\alpha \log \frac{X^0 \cdot S^0}{\varepsilon})$ iterations and the complexity of each iteration is bounded above by $O(\alpha\beta^3, \alpha^2\beta^2, \alpha^2)$ being $\alpha = 3n + 2nd(1+\log r)$ and $\beta = p$, the dimension of the dual matrix variable S_p.*

The reader may observe that this result is mainly of theoretical interest because the bound is based on general results on primal-dual algorithms, such as the modification of Kouleai and Terlaki [33] to the Mehrota type algorithm [38] applied to SDP problems. Nevertheless, it is important to realize that it states an important difference with respect to any other known result in the area of continuous location where convergence results, when available, are only proven limit of sequences and never in finite number of steps nor accuracy ensured. In this case, one can ensure a prespecified accuracy of the solution in a known number of iterations.

3 Constrained ordered median problems

This section extends the above results to constrained location problems. Therefore, we address now the restricted case of Problem (1). The reader may observe that this extension is rather interesting from the application point of view since in most occasions location problems are not unconstrained. Nevertheless, it is far from trivial as can be seen by the few exact algorithms that exist in the literature for continuous ordered median location problems with additional constraints. (The interested reader is referred to [19,20] and the references therein for further details.)

In order to present this approach, we need to describe our framework. Let $\{g_1, \ldots, g_l\} \subset \mathbb{R}[x]$ be real polynomials and $\mathbf{K} := \{x \in \mathbb{R}^d : g_j(x) \ge 0, \ j =$

$1, \ldots, l\}$ a basic closed, compact semialgebraic set with nonempty interior satisfying the Archimedean property. Recall that the Archimedean property is equivalent to imposing that for some $M > 0$ the quadratic polynomial $u(x) = M - \sum_{i=1}^{d} x_i^2$ has a representation on $\mathbf{K}$ as $u = \sigma_0 + \sum_{j=1}^{\ell} \sigma_j g_j$, for some $\{\sigma_0, \ldots, \sigma_l\} \subset \mathbb{R}[x]$ being each σ_j sum of squares [48]. We remark that the assumption on the Archimedean property is not restrictive at all, since any semialgebraic set $\mathbf{K} \subseteq \mathbb{R}^d$ for which is known that $\sum_{i=1}^{d} x_i^2 \leq M$ holds for some $M > 0$ and for all $x \in \mathbf{K}$, admits a new representation $\mathbf{K}' = \mathbf{K} \cup \{x \in \mathbb{R}^d : g_{l+1}(x) := M - \sum_{i=1}^{d} x_i^2 \geq 0\}$ that trivially verifies the Archimedean property. In our framework the compactness assumption which is usually assumed in location analysis implies that this condition always holds.

In this framework we assume that the domain $\mathbf{K}$ is compact and has nonempty interior. We observe that we can extend the results in Sect. 2 to a broader class of convex constrained problems.

In order to do that we need to introduce some notations. Let $\boldsymbol{\kappa} = (\kappa_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta z^\gamma v^\delta w^\zeta u^\eta y^\theta)$ of $\mathbb{R}[x, z, v, w, u, y]$ (with $\alpha = (\beta, \gamma, \delta, \zeta, \eta, \theta) \in \mathbb{N}^d \times \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^{n\times d} \times \mathbb{N}^{n\times d}$). Let $D = 3n + (2n+1)d$ denotes the dimension of the space of variables. Define $\Upsilon = (x, z, v, w, u, y)$ to be the vector of indeterminates so that $\Upsilon^\alpha = x^\beta z^\gamma v^\delta w^\zeta u^\eta y^\theta$. For any integer N consider the monomial vector

$$[\Upsilon^N] = [(x, z, v, w, u, y)^N] = [1, x_1, \ldots x_d, z_1, \ldots z_n, \ldots, y_{nd}, x_1^2, x_1 x_2, \ldots, y_{nd}^N]^t.$$

Then, $[\Upsilon^N][\Upsilon^N]^t$ is a square matrix and we write

$$[\Upsilon^N][\Upsilon^N]^t = \sum_{0 \leq |\alpha| \leq 2N} A_\alpha^0 \Upsilon^\alpha$$

for some symmetric 0/1 matrices A_α^0. Here, for a vector α, $|\alpha|$ stands for the sum of its components.

For any sequence, $\boldsymbol{\kappa} = (\kappa_\alpha)_{\alpha \in \mathbb{N}^D} \subset \mathbb{R}$, indexed in the canonical monomial basis $\mathcal{B}$, let $\mathrm{L}_{\boldsymbol{\kappa}} : \mathbb{R}[\Upsilon] \to \mathbb{R}$ be the linear functional defined, for any $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha \Upsilon^\alpha \in \mathbb{R}[\Upsilon]$, as $\mathrm{L}_{\boldsymbol{\kappa}}(f) := \sum_\alpha f_\alpha \kappa_\alpha$.

The moment matrix $\mathrm{M}_N(\boldsymbol{\kappa})$ of order N associated with $\boldsymbol{\kappa}$, has its rows and columns indexed by (Υ^α) and $\mathrm{M}_N(\boldsymbol{\kappa})(\alpha, \alpha') := \mathrm{L}_{\boldsymbol{\kappa}}(\Upsilon^{\alpha+\alpha'}) = \kappa_{\alpha+\alpha'}$, for $|\alpha|, |\alpha'| \leq N$. Therefore,

$$\mathrm{M}_N(\boldsymbol{\kappa}) = \sum_{0 \leq |\alpha| \leq 2N} A_\alpha^0 \kappa_\alpha$$

Note that the moment matrix of order N has dimension $\binom{D+N}{D} \times \binom{D+N}{D}$ and that there are $\binom{D+2N}{D}$ $\boldsymbol{\kappa}_\alpha$ variables.

For $g \in \mathbb{R}[\Upsilon]$ $(= \sum_{\nu \in \mathbb{N}^M} g_\nu \Upsilon^\nu)$, the localizing matrix $\mathrm{M}_N(g\boldsymbol{\kappa})$ of order N associated with $\boldsymbol{\kappa}$ and g, has its rows and columns indexed by (Υ^α) and $\mathrm{M}_N(g\boldsymbol{\kappa})(\alpha, \alpha') := \mathrm{L}_{\boldsymbol{\kappa}}(\Upsilon^{\alpha+\alpha'} g(\Upsilon)) = \sum_\nu g_\nu \kappa_{\nu+\alpha+\alpha'}$, for $|\alpha|, |\alpha'| \leq N$. Therefore,

$$\mathrm{M}_N(g\boldsymbol{\kappa}) = \sum_{0\le|\alpha|\le 2N} A_\alpha^g \kappa_\alpha,$$

for some symmetric 0/1 matrices A_α^g that depend on the polynomial g. Also for convenience, we shall denote by $A_{e_i}^g$ the matrix associated with κ_{e_i} the moment variable linked to the monomial $x^{e_i} = x_i^1$. (The interested reader is referred to [34,35] for further details on the moment approach applied to global optimization.)

The following result states sufficient conditions ensuring that the constrained ordered median problem can be solved similarly to the unconstrained one using an alternative SDP approach.

Theorem 4 *Consider the restricted problem:*

$$\min_{x\in\mathbf{K}\subset\mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau. \tag{28}$$

Assume that the hypothesis of Theorem 2 holds. In addition, any of the following conditions holds:

1. *$g_i(x)$ are concave for $i = 1, \ldots, \ell$ and $-\sum_{i=1}^{\ell} \mu_i \nabla^2 g_i(x) \succ 0$ for each dual pair (x, μ) of the problem of minimizing any linear functional $c^t x$ on $\mathbf{K}$ (Positive Definite Lagrange Hessian (PDLH)).*
2. *$g_i(x)$ are sos-concave on $\mathbf{K}$ for $i = 1, \ldots, \ell$ or $g_i(x)$ are concave on $\mathbf{K}$ and strictly concave on the boundary of $\mathbf{K}$ where they vanish, i.e. $\partial\mathbf{K} \cap \partial\{x \in \mathbb{R}^d : g_i(x) = 0\}$, for all $i = 1, \ldots, \ell$.*
3. *$g_i(x)$ are strictly quasi-concave on $\mathbf{K}$ for $i = 1, \ldots, \ell$.*

Then, there exists a constructive finite dimension embedding, which only depends on τ and g_i, $i = 1, \ldots, \ell$, such that (28) is a semidefinite problem.

Proof The unconstrained version of Problem (28) can be equivalently written as a SDP using the result in Theorem 2. Therefore, it remains to prove that under the conditions 1, 2 or 3 the constraint set $x \in \mathbf{K}$ is also exactly represented as a finite number of semidefinite constraints or equivalently that it is semidefinite representable (SDr).

Let us begin with condition 1. Consider the system of linear matrix inequalities:

$$A_0^{(k)} + \sum_{i=1}^{\ell} A_{e_i}^{g_k} x_i + \sum_{1\le\alpha\le 2N} A_\alpha^{g_k} \kappa_\alpha \succeq 0, \qquad k = 0, \ldots, \ell. \tag{29}$$

Under the hypothesis of condition 1, the set $\mathbf{K}$ satisfies the Putinar-Prestel's Bounded Degree Nonnegative Representation property (PP-BDNR), see [26, Theorem 6]. This condition ensures that there exists a finite N such that the set

$$\hat{S}_N = \{(x, \kappa) : \text{ satisfying inequalites (29)}\}$$

projects via the x coordinate onto the set $\mathbf{K}$. Hence, an exact lifted representation of Problem (28) is the one provided by Theorem 2 augmented with the additional linear matrix inequalities in (29).

Let us assume now that condition 2 holds. Consider the set

$$\hat{S}_N = \{(x,\kappa) : \mathrm{M}_N(\boldsymbol{\kappa}) \succeq 0,\ L_{\boldsymbol{\kappa}}(g_i) \geq 0,\ i = 1\ldots,\ell,\\ L_{\boldsymbol{\kappa}}(x_j) = x_j,\ j = 1,\ldots,d,\ \kappa_0 = 1\}.$$

Under our hypothesis, Theorem 11.11 in [35] ensures that there exists a finite N such that $\hat{S}_N$ projects via the x variables onto the set $\mathbf{K}$. Hence, we obtain another lifted exact SDP formulation for Problem (28) using the formulation induced by Theorem 2 augmented with the inequalities $\mathrm{M}_N(\boldsymbol{\kappa}) \succeq 0,\ L_{\boldsymbol{\kappa}}(g_i) \geq 0,\ i = 1\ldots,\ell,\ L_{\boldsymbol{\kappa}}(x_j) = x_j,\ j = 1,\ldots,d,\ \kappa_0 = 1$.

Finally, let us consider the case in condition 3. If g_i are strictly quasi-concave on $\mathbf{K}$, Proposition 10 in [26] implies that one can find some new polynomials $-p_i$ that have positive definite Hessian in $\mathbf{K}$. Let us denote $P := \{x \in \mathbb{R}^d : p_i(x) \geq 0,\ i = 1,\ldots,\ell\}$. Thus, in some open set U containing $\mathbf{K}$ it holds $P \cap U = \mathbf{K}$.

Next, define the set

$$\hat{S}_N = \{(x,\boldsymbol{\kappa}) : \text{ satisfying inequalites (30)–(33)}\}$$

where the set of linear matrix inequalities (30)–(33) are given by:

$$A_0^{(k)} + \sum_{i=1}^{\ell} A_{e_i}^{(k)} x_i + \sum_{1\leq\alpha\leq 2N} A_\alpha^{p_k}\kappa_\alpha \succeq 0, k = 0,\ldots,\ell \tag{30}$$

$$L_{\boldsymbol{\kappa}}(p_k) \geq 0, k = 0,\ldots,\ell \tag{31}$$

$$L_{\boldsymbol{\kappa}}(x_j) = x_j, j = 1,\ldots,d \tag{32}$$

$$\kappa_0 = 1. \tag{33}$$

Under the hypothesis of condition 3, Theorem 24 in [26] ensures that there exists a finite N such that $\hat{S}_N$ projects via the x variables onto $\mathbf{K}$. Hence, we obtain the third lifted exact SDP formulation for Problem (28) using the formulation induced by Theorem 2 augmented with the inequalities (30)–(33).

We observe that according to Theorem 29 in [26], since we assume the Archimedean property holds in all these cases, N can be bounded above by some finite constant that only depends on the polynomials $g_i,\ i = 1,\ldots,\ell$. □

The above result extends the SDP analysis to some classes of constrained, convex ordered median problems. However, it can be extended further, with some different tools borrowed from the Theory of Moments [35], to more general cases. We shall finish this section with another convergence result applicable to the case of non-convex constrained location problems. Again, let $\{g_1,\ldots,g_l\} \subset \mathbb{R}[x]$ and $\mathbf{K} := \{x \in \mathbb{R}^d : g_k(x) \geq 0,\ k = 1,\ldots,\ell\}$ a basic, compact, closed semialgebraic set satisfying the Archimedean property, with nonempty interior and such that $\mathbf{K}$ does not satisfy the hypothesis of Theorem 4, in particular some of the g_j may not be concave.

Now, we can prove a convergence result that allows us to approximate, up to any degree of accuracy, the solution of the class of problems defined in (28) when the hypothesis of Theorem 4 fails. Let $\xi_k := \lceil (\deg g_k)/2 \rceil$ where $\{g_1, \dots, g_\ell\}$ are the polynomial constraints that define $\mathbf{K}$. For $N \geq N_0 := \max\{\max_{k=1,\dots,\ell} \xi_k, 1\}$, we introduce the following hierarchy of semidefinite programs:

$$
\begin{aligned}
(\mathbf{Q}_N) : \min \quad & \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i && (34)\\
s.t. \quad & v_i + w_k \geq \lambda_k z_i, \quad \forall i, k = 1, ..., n, && (35)\\
& y_{ij} - x_j + a_{ij} \geq 0, \quad \forall i = 1, ..., n, \ j = 1, ..., d. && (36)\\
& y_{ij} + x_j - a_{ij} \geq 0, \quad \forall i = 1, ..., n, \ j = 1, ..., d. &&\\
& y_{ij}^r \leq u_{ij}^s z_i^{r-s}, \quad \forall i = 1, ..., n, \ j = 1, ..., d, && (37)\\
& \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \leq z_i, \quad \forall i = 1, ..., n, && (38)\\
& \mathrm{M}_N(\boldsymbol{\kappa}) \succeq 0, && (39)\\
& \mathrm{M}_{N-\xi_k}(g_k, \boldsymbol{\kappa}) \succeq 0, \quad k = 1, \dots, \ell, && (40)\\
& L_\kappa(x_j) = x_j, \quad j = 1, \dots, d, &&\\
& L_\kappa(z_i) = z_i, \quad i = 1, \dots, n, &&\\
& L_\kappa(v_i) = v_i, \quad i = 1, \dots, n, &&\\
& L_\kappa(w_i) = w_i, \quad i = 1, \dots, n, &&\\
& L_\kappa(u_{ij}) = u_{ij}, \quad i = 1, \dots, n, \ j = 1, \dots, d, &&\\
& L_\kappa(y_{ij}) = y_{ij}, \quad i = 1, \dots, n, \ j = 1, \dots, d, &&\\
& \kappa_0 = 1 &&\\
& u_{ij} \geq 0, \quad \forall\, i = 1, ..., n, \ j = 1, \dots, d. && (41)
\end{aligned}
$$

with optimal value denoted $\min \mathbf{Q}_N$.

Theorem 5 *Consider ρ_λ defined as the optimal value of the problem:*

$$\rho_\lambda = \min_{x \in \mathbf{K} \subset \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau. \tag{42}$$

Then, with the notation above:

(a) $\min \mathbf{Q}_N \uparrow \rho_\lambda$ *as* $N \to \infty$.
(b) Let $\boldsymbol{\kappa}^N$ *be an optimal solution of Problem* $(\mathbf{Q}_N)$. *If*

$$\operatorname{rank} \mathrm{M}_N(\boldsymbol{\kappa}^N) = \operatorname{rank} \mathrm{M}_{N-N_0}(\boldsymbol{\kappa}^r) = \vartheta$$

then $\min \mathbf{Q}_N = \rho_\lambda$ *and one may extract* ϑ *points*

$$(x^*(i), z^*(i), v^*(i), w^*(i), u^*(i), y^*(i))_{i=1}^{\vartheta} \subset \mathbf{K},$$

all global minimizers of Problem (42).

Proof First of all, we observe that an optimal solution of Problem (42) does exist by the compactness assumption on $\mathbf{K}$. Moreover, the convergence of the semidefinite sequence of problems ($\mathbf{Q}_N$) follows from a result by Lasserre [35, Theorem 5.6] that it is applied here to the SDP problem (2–8) on the closed semialgebraic set $\mathbf{K}$. The second assertion on the rank condition, for extracting optimal solutions, follows from applying [35, Theorem 5.7] to Problem ($\mathbf{Q}_N$). □

4 Computational experiments

A series of computational experiments have been performed in order to evaluate the behavior of the proposed methodology. Programs have been coded in MATLAB R2010b and executed in a PC with an Intel Core i7 processor at 2x 2.93 GHz and 16 GB of RAM. The problems have been solved by calling SeDuMi 3.01 [51], using either the SDP or the SOCP internal algorithm implemented in this solver. Therefore, our CPU times (`Time(Ave)`) and accuracies/gaps (`Gap(Ave)`) reported in the tables are referred to this solver.

We run the algorithm for several well-known continuous single facility convex ordered location problems: Weber, center, k-center and general ordered median problem with random non-increasing monotone lambda. For each of them, we obtain the CPU times for computing solutions as well as the accuracy given by the solver SeDuMi 3.01 (see Tables 1, 2, 3, 4, that for the ease of presentation are collected at the end of the paper). The reader may observe that since this approach is exact, this accuracy (*gap*) is the one reported by the solver due to its internal precision. In addition, to illustrate the application of the result in Theorem 5, we also report results on a problem which consists of minimizing the range of distances in $\mathbb{R}^3$ with two additional non-convex constraints. In this case, we include running times and gap with respect to upper bounds obtained with the battery of functions in `optimset` of MATLAB which only provide approximations on the exact solutions (optimality cannot be certified).

In this last case, in order to compute the accuracy of an obtained solution, we use the following measure for the error (see [53]):

$$\epsilon_{\text{obj}} = \frac{|\text{the optimal value of the SDP } - \text{fopt}|}{\max\{1, \text{fopt}\}}, \tag{43}$$

where fopt is the approximated optimal value obtained with the functions in `optimset`.

The reader may note that in this case we solve relaxed problems that give lower bounds. Therefore, the gap of our lower bounds is computed with respect to upper bounds (the solution reported by `optimset` is a heuristic solution) which implies that actually may be even better than the one reported (see Table 5).

We have organized our computational experiments in four different problems types, namely Weber, center, k-centrum and general λ; and for each of them we compare the performance of our model using the SDP or the SOCP approach. Our test problems are set of points randomly generated on the [0, 10000] hypercubes of the d-dimensional space, $d = 2, 3, 10$. For Weber, center and k-centrum problems, we could solve, at

Table 1 Computational results for Weber problem with different norms and different dimensions

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.20	0.06	10^{-8}	$<10^{-8}$	0.28	0.06	10^{-8}	$<10^{-8}$	0.86	0.09	$<10^{-8}$	$<10^{-8}$
	100	1.71	0.27	10^{-8}	$<10^{-8}$	3.16	0.40	10^{-8}	$<10^{-8}$	10.89	8.77	10^{-8}	$<10^{-8}$
	500	10.78	15.34	10^{-8}	$<10^{-8}$	15.84	22.32	10^{-8}	$<10^{-8}$	51.23	1883.8	$<10^{-8}$	$<10^{-8}$
	1,000	21.22	128.53	10^{-8}	$<10^{-8}$	30.67	179.72	10^{-8}	$<10^{-8}$	103.17	24170.12	10^{-8}	$<10^{-8}$
	5000	103.50	57013.83	10^{-8}	$<10^{-8}$	178.50	NaN	10^{-8}	NaN	566.64	NaN	10^{-8}	NaN
	10,000	210.22	NaN	10^{-8}	NaN	455.05	NaN	10^{-8}	NaN	1330.36	NaN	10^{-8}	NaN
2	10	0.06	0.04	$<10^{-8}$	$<10^{-8}$	0.12	0.04	$<10^{-8}$	$<10^{-8}$	0.46	0.04	$<10^{-8}$	$<10^{-8}$
	100	0.40	0.05	$<10^{-8}$	$<10^{-8}$	0.99	0.07	$<10^{-8}$	$<10^{-8}$	4.63	0.04	$<10^{-8}$	$<10^{-8}$
	500	1.50	0.12	$<10^{-8}$	$<10^{-8}$	5.83	0.14	$<10^{-8}$	$<10^{-8}$	21.63	0.14	$<10^{-8}$	$<10^{-8}$
	1,000	3.27	0.42	$<10^{-8}$	$<10^{-8}$	11.08	0.48	$<10^{-8}$	$<10^{-8}$	44.04	0.36	$<10^{-8}$	$<10^{-8}$
	5,000	17.06	19.90	$<10^{-8}$	$<10^{-8}$	58.16	22.09	$<10^{-8}$	$<10^{-8}$	218.82	15.59	$<10^{-8}$	$<10^{-8}$
	10,000	33.19	137.81	$<10^{-8}$	$<10^{-8}$	118.91	162.98	$<10^{-8}$	$<10^{-8}$	455.30	107.06	$<10^{-8}$	$<10^{-8}$
3	10	0.19	0.06	10^{-8}	$<10^{-8}$	0.31	0.14	10^{-8}	10^{-8}	0.99	0.54	10^{-8}	5×10^{-8}
	100	1.88	1.77	10^{-8}	2×10^{-6}	3.60	3.66	10^{-8}	2×10^{-5}	12.71	43.54	10^{-8}	9×10^{-8}
	500	10.82	24.44	10^{-8}	8×10^{-4}	17.87	124.97	10^{-8}	8×10^{-4}	57.91	3362.08	10^{-8}	5×10^{-5}
	1,000	21.73	55.03	10^{-8}	7×10^{-4}	33.99	279.74	10^{-8}	2×10^{-3}	118.11	NaN	10^{-8}	NaN
	5,000	110.87	NaN	10^{-8}	NaN	181.17	NaN	10^{-8}	NaN	646.46	NaN	10^{-8}	NaN
	10,000	245.66	NaN	10^{-8}	NaN	477.38	NaN	10^{-8}	NaN	1616.26	NaN	10^{-8}	NaN
3.5	10	0.33	0.12	10^{-8}	$<10^{-8}$	0.47	0.16	10^{-8}	2×10^{-8}	1.75	0.43	10^{-8}	$<10^{-8}$

Table 1 continued

τ	n	d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
		SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
	100	3.87	3.31	10^{-8}	4×10^{-6}	5.44	5.44	10^{-8}	9×10^{-8}	19.71	124.31	10^{-8}	2×10^{-7}
	500	18.62	49.23	10^{-8}	3×10^{-3}	26.99	242.89	10^{-8}	3×10^{-3}	92.64	16092.68	10^{-8}	10^{-3}
	1,000	37.06	190.64	10^{-8}	2×10^{-3}	51.50	799.74	10^{-8}	2×10^{-3}	192.77	21409.00	10^{-8}	7×10^{-4}
	5,000	280.27	NaN	10^{-8}	NaN	304.94	NaN	10^{-8}	NaN	1178.17	NaN	10^{-8}	NaN
	10,000	964.18	NaN	10^{-8}	NaN	872.29	NaN	10^{-8}	NaN	2431.58	NaN	10^{-8}	NaN

Table 2 Computational results for the Center problem with different norms and different dimensions

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.22	0.05	10^{-6}	$<10^{-8}$	0.39	0.21	$<10^{-8}$	4×10^{-5}	1.02	0.10	10^{-4}	2×10^{-2}
	100	2.13	0.20	$<10^{-8}$	$<10^{-8}$	8.75	0.53	$<10^{-8}$	4×10^{-5}	43.28	9.03	2×10^{-5}	2×10^{-2}
	500	12.59	8.20	$<10^{-8}$	$<10^{-8}$	63.16	38.33	$<10^{-8}$	9×10^{-5}	237.99	1435.45	8×10^{-6}	8×10^{-3}
	1,000	27.11	56.80	$<10^{-8}$	$<10^{-8}$	115.35	315.47	$<10^{-8}$	5×10^{-5}	327.04	NaN	6×10^{-6}	NaN
	5,000	150.34	26786.61	10^{-8}	$<10^{-8}$	357.49	NaN	$<10^{-8}$	NaN	1231.78	NaN	5×10^{-6}	NaN
	10,000	371.39	NaN	$<10^{-8}$	NaN	1297.23	NaN	$<10^{-8}$	NaN	2762.51	NaN	7×10^{-6}	NaN
2	10	0.11	0.03	5×10^{-8}	5×10^{-6}	0.17	0.04	10^{-8}	6×10^{-5}	0.44	0.04	2×10^{-5}	10^{-2}
	100	1.34	0.05	10^{-7}	7×10^{-6}	2.09	0.05	10^{-8}	4×10^{-6}	7.68	0.06	2×10^{-4}	8×10^{-3}
	500	8.80	0.09	8×10^{-8}	6×10^{-6}	13.69	0.10	10^{-8}	10^{-5}	50.25	0.14	10^{-5}	5×10^{-3}
	1,000	20.85	0.11	5×10^{-7}	5×10^{-6}	30.27	0.16	10^{-8}	10^{-5}	115.64	0.26	10^{-5}	6×10^{-3}
	5,000	119.90	0.68	10^{-6}	4×10^{-6}	212.21	0.78	10^{-8}	2×10^{-4}	912.44	1.37	4×10^{-3}	5×10^{-3}
	10,000	287.13	1.31	3×10^{-6}	10^{-5}	467.08	1.39	10^{-8}	8×10^{-5}	1510.41	2.48	8×10^{-3}	5×10^{-3}
3	10	0.23	0.07	2×10^{-7}	$<10^{-8}$	0.37	0.23	10^{-8}	2×10^{-5}	1.08	0.09	10^{-4}	7×10^{-3}
	100	2.32	0.28	$<10^{-8}$	$<10^{-8}$	7.48	0.48	$<10^{-8}$	3×10^{-7}	37.66	9.58	10^{-5}	10^{-2}
	500	14.47	14.78	10^{-8}	$<10^{-8}$	52.27	34.85	2×10^{-7}	4×10^{-6}	209.46	1446.29	6×10^{-6}	5×10^{-3}
	1,000	28.93	110.24	10^{-8}	$<10^{-8}$	119.96	297.47	6×10^{-7}	4×10^{-5}	293.48	NaN	2×10^{-5}	NaN
	5,000	160.96	19057.62	10^{-8}	$<10^{-8}$	456.11	NaN	10^{-4}	NaN	1223.41	NaN	2×10^{-4}	NaN
	10,000	434.79	NaN	$<10^{-8}$	NaN	6829.70	NaN	3×10^{-5}	NaN	2663.15	NaN	3×10^{-4}	NaN
3.5	10	0.33	0.06	$<10^{-8}$	10^{-8}	0.56	0.07	6×10^{-6}	5×10^{-5}	1.80	0.17	5×10^{-5}	10^{-2}

Table 2 continued

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
	100	4.47	0.38	10^{-7}	$<10^{-8}$	13.72	0.83	4×10^{-7}	2×10^{-5}	68.91	19.60	2×10^{-5}	6×10^{-3}
	500	21.93	19.68	2×10^{-8}	$<10^{-8}$	90.65	58.54	2×10^{-7}	9×10^{-6}	373.21	3658.09	2×10^{-6}	6×10^{-3}
	1,000	44.82	246.90	2×10^{-8}	$<10^{-8}$	179.48	963.05	2×10^{-5}	6×10^{-7}	603.41	NaN	10^{-5}	NaN
	5,000	244.04	NaN	2×10^{-8}	NaN	551.16	NaN	2×10^{-6}	NaN	2279.93	NaN	2×10^{-4}	NaN
	10,000	510.25	NaN	2×10^{-8}	NaN	2618.87	NaN	2×10^{-5}	NaN	4814.26	NaN	6×10^{-5}	NaN

Table 3 Computational results for the $0.5n$-centrum problem with different norms and different dimensions

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.29	0.11	10^{-8}	$<10^{-8}$	0.42	0.10	$<10^{-8}$	10^{-8}	1.01	0.13	2×10^{-4}	2×10^{-4}
	100	4.67	0.53	$<10^{-8}$	$<10^{-8}$	8.01	1.18	$<10^{-8}$	$<10^{-8}$	30.45	17.07	3×10^{-8}	5×10^{-8}
	500	39.92	32.40	10^{-8}	$<10^{-8}$	49.26	96.39	$<10^{-8}$	$<10^{-8}$	205.37	2794.70	2×10^{-8}	10^{-8}
	1,000	68.11	214.23	10^{-8}	$<10^{-8}$	109.52	659.33	$<10^{-8}$	$<10^{-8}$	437.95	53531.38	2×10^{-8}	$<10^{-8}$
	5,000	476.95	NaN	$<10^{-8}$	NaN	668.08	NaN	$<10^{-8}$	NaN	4738.70	NaN	2×10^{-8}	NaN
	10,000	1242.57	NaN	10^{-8}	NaN	2016.01	NaN	$<10^{-8}$	NaN	15348.57	NaN	2×10^{-8}	NaN
2	10	0.13	0.07	$<10^{-8}$	$<10^{-8}$	0.18	0.05	$<10^{-8}$	$<10^{-8}$	0.45	0.05	8×10^{-5}	4×10^{-4}
	100	1.36	0.13	$<10^{-8}$	$<10^{-8}$	2.03	0.10	$<10^{-8}$	$<10^{-8}$	7.56	0.10	10^{-8}	$<10^{-8}$
	500	8.40	0.68	$<10^{-8}$	$<10^{-8}$	15.63	0.48	$<10^{-8}$	$<10^{-8}$	53.87	0.47	$<10^{-8}$	$<10^{-8}$
	1,000	22.38	3.71	$<10^{-8}$	$<10^{-8}$	30.90	2.13	$<10^{-8}$	$<10^{-8}$	108.22	1.99	$<10^{-8}$	$<10^{-8}$
	5,000	128.18	397.93	$<10^{-8}$	$<10^{-8}$	195.56	293.61	$<10^{-8}$	$<10^{-8}$	815.34	228.04	$<10^{-8}$	$<10^{-8}$
	10,000	337.17	3112.30	$<10^{-8}$	$<10^{-8}$	460.98	2129.80	$<10^{-8}$	$<10^{-8}$	2373.23	2545.92	$<10^{-8}$	$<10^{-8}$
3	10	0.30	0.10	10^{-8}	$<10^{-8}$	0.42	0.11	$<10^{-8}$	2×10^{-8}	1.09	0.13	3×10^{-5}	4×10^{-7}
	100	5.65	0.66	10^{-8}	$<10^{-8}$	10.20	1.46	$<10^{-8}$	$<10^{-8}$	40.08	21.77	2×10^{-7}	$<10^{-8}$
	500	50.36	41.43	10^{-8}	$<10^{-8}$	72.35	125.78	$<10^{-8}$	$<10^{-8}$	225.13	2331.94	4×10^{-8}	$<10^{-8}$
	1,000	100.17	418.97	10^{-8}	$<10^{-8}$	145.24	774.01	$<10^{-8}$	$<10^{-8}$	463.74	NaN	4×10^{-8}	NaN
	5,000	582.84	NaN	2×10^{-8}	NaN	894.95	NaN	$<10^{-8}$	NaN	4067.13	NaN	2×10^{-8}	NaN
	10,000	1715.00	NaN	10^{-8}	NaN	2565.21	NaN	2×10^{-8}	NaN	13649.88	NaN	5×10^{-8}	NaN
3.5	10	0.44	0.11	6×10^{-8}	$<10^{-8}$	0.60	0.12	$<10^{-8}$	$<10^{-8}$	1.81	0.26	2×10^{-4}	2×10^{-6}

Table 3 continued

τ	n	d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
		SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
	100	10.90	1.58	10^{-8}	$<10^{-8}$	16.97	3.54	$<10^{-8}$	$<10^{-8}$	60.45	61.05	4×10^{-8}	10^{-8}
	500	80.28	133.69	2×10^{-8}	$<10^{-8}$	124.50	347.44	2×10^{-8}	$<10^{-8}$	379.20	6691.80	4×10^{-8}	$<10^{-8}$
	1,000	171.62	869.75	2×10^{-8}	$<10^{-8}$	252.79	2367.65	2×10^{-8}	10^{-8}	852.59	211506.04	4×10^{-8}	$<10^{-8}$
	5,000	1033.28	NaN	2×10^{-8}	NaN	1700.71	NaN	2×10^{-8}	NaN	8510.86	NaN	6×10^{-8}	NaN
	10,000	2345.25	NaN	2×10^{-8}	NaN	4682.55	NaN	2×10^{-8}	NaN	27723.99	NaN	4×10^{-8}	NaN

Table 4 Results for convex ordered median problem with general λ, different norms and dimensions

		d											
		2				3				10			
		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)		Time(Ave)		Gap(Ave)	
τ	n	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP	SDP	SOCP
1.5	10	0.24	0.19	$<10^{-8}$	$<10^{-8}$	0.40	0.09	$<10^{-8}$	$<10^{-8}$	1.19	0.14	3×10^{-6}	10^{-5}
	100	4.03	1.33	10^{-8}	$<10^{-8}$	6.73	1.88	10^{-8}	$<10^{-8}$	22.22	18.46	$<10^{-8}$	$<10^{-8}$
	500	159.42	75.98	10^{-8}	$<10^{-8}$	190.77	141.67	10^{-8}	$<10^{-8}$	380.99	1322.44	$<10^{-8}$	3×10^{-8}
	1,000	1270.76	NaN	10^{-7}	NaN	1730.61	NaN	2×10^{-8}	NaN	2379.03	NaN	$<10^{-8}$	NaN
2	10	0.14	0.04	3×10^{-8}	10^{-8}	0.18	0.06	10^{-8}	10^{-8}	0.62	0.06	3×10^{-5}	2×10^{-6}
	100	5.11	0.31	3×10^{-6}	10^{-8}	6.94	0.37	2×10^{-8}	10^{-8}	17.07	0.29	3×10^{-8}	$<10^{-8}$
	500	427.54	12.88	3×10^{-6}	10^{-8}	443.47	15.29	3×10^{-6}	$<10^{-8}$	1092.70	11.35	5×10^{-7}	$<10^{-8}$
	1,000	2079.97	100.51	3×10^{-6}	10^{-8}	7702.03	127.81	2×10^{-6}	$<10^{-8}$	9235.59	81.98	7×10^{-7}	$<10^{-8}$
3	10	0.51	0.10	5×10^{-8}	2×10^{-8}	0.72	0.16	10^{-7}	6×10^{-8}	1.89	0.28	4×10^{-6}	3×10^{-6}
	100	64.14	3.49	5×10^{-6}	10^{-6}	58.10	5.03	10^{-6}	5×10^{-7}	152.45	46.06	10^{-6}	2×10^{-7}
	500	1532.17	272.56	2×10^{-4}	10^{-4}	2269.39	413.24	3×10^{-5}	7×10^{-5}	7950.09	4100.40	9×10^{-6}	2×10^{-5}
	1,000	4546.73	2080.30	3×10^{-4}	10^{-4}	5678.17	NaN	9×10^{-5}	NaN	18011.79	29734.01	3×10^{-5}	10^{-5}
3.5	10	0.63	0.16	10^{-6}	10^{-8}	1.48	0.19	10^{-5}	4×10^{-8}	7.05	0.34	2×10^{-6}	2×10^{-5}
	100	33.32	10.01	10^{-6}	4×10^{-6}	302.76	15.03	9×10^{-5}	3×10^{-6}	596.20	163.09	2×10^{-4}	7×10^{-8}
	500	1555.08	438.34	5×10^{-6}	3×10^{-5}	2774.06	817.91	4×10^{-4}	4×10^{-4}	8705.00	19654.02	$\times10^{-4}$	2×10^{-4}
	1,000	7625.95	NaN	2×10^{-5}	NaN	7681.10	NaN	6×10^{-4}	NaN	18845.92	NaN	2×10^{-4}	NaN

Table 5 Computational results for the range problem with two nonconvex constraints

		d	
		3	
τ	n	Time(Ave)	Gap(Ave)
2	10	0.46	0.00001623
	100	9.45	0.00457982
	500	80.56	0.00030263
	1,000	204.96	0.00094492

least with one of the approaches (SDP or SOCP), instances with at least 10,000 points for the different ℓ_τ-norms, $\tau = 1.5, 2, 3, 3.5$. The general case with random lambda weights is harder and we only solved in all cases instances up to 1,000 points.

Our goal is to present the results organized per problem type, framework space ($\mathbb{R}^d$, $d = 2, 3, 10$), norm (ℓ_τ, $\tau = 1.5, 2, 3, 3.5$) and distinguishing whether we apply SDP or SOCP algorithms (`SDP` or `SOCP` columns). Tables 1, 2 and 3 report our results on the problems of minimizing the weighted sum of distances (Weber), the maximum distance (center) and the sum of the $n/2$ largest distances ($n/2$-centrum). In all cases, the accuracy and resolutions times needed for the solver are rather good, even for 10,000 points in dimension 10 and rather complicated norms (e.g. $\ell_{3.5}$). In our tables we have used the notation " $< 10^{-a}$ " to indicate an accuracy greater than 10^{-a}. The reader can see that the hardest type is the k-centrum. In this problem type CPU times increase one order of magnitude because the structure of the problem does not allow to reduce the size of the formulation.

Table 4 reports our results on the general ordered median problem with non-increasing monotone lambda weights. For this family of problems we could solve with our general formulation and the SDP approach, in all cases, problem sizes of 1,000 points. Accuracy is rather good and the bottleneck here is the size of the formulation to be handle since the fact that all lambda are non-null makes it impossible to simplify the representation.

Comparing the performance of the SDP versus SOCP representations of our model, we observe that if the number of points is less than or equal to $n = 100$, SOCP approach results in better CPU times for all problem types, dimension and norm as expected by the theoretical worst-case complexity of SOCP. If the number of demand points $n = 500$ then SOCP is better that SDP in dimensions $d = 2, 3$. For more that 500 demand points, with the exception of norm ℓ_2, SDP approach gives better CPU times and gaps than SOCP in all problems. Moreover, SOCP fails to properly solve, either due to numerical errors or lack of memory, most instances with $n = 5,000, 10,000$ and several with $n = 1,000$; depending on the dimension d and problem type. This is shown in the tables with the symbol `NaN` (*Not a Number*) that is flag reported by the solver. It is important to remark that our comparison is based on a specific solver, we have chosen SeDuMi since it has the feature to solve both SDP and SOCP. (SDPT3 [32] performs similarly.) The explanation for a better performance of the SDP reformulation versus the SOCP one, for large size instances, may be due to the fact that, as reported for instance by Mittelmann [36], SeDuMi is able to exploit sparsity patterns in structured problems for SDP problems. (The reader may observe that location problems exhibit well-known sparsity patterns.) The better implementation of SeDuMi for SDP than for SOCP problems seems to compensate, for large size instances, the best worst case complexity of the SOCP algorithm (see Tables 1, 2, 3, 4). Needless to say that our results may depend on the solver and the internal algorithms that are implemented and one could obtain also better results with the SOCP reformulation, even for larger size instances, using a specialized SOCP solver.

Finally, we also report in Table 5, for the sake of illustration, an example of application of the result in Theorem 5. This problem consists of the minimization of the difference between the maximum and minimum distances of a number of demand points (n ranging between 10 and 1,000) with respect to a solution point that must belong to a non-convex feasible region defined by the following two non-convex constraints $x_1^2 - 2x_2^2 - 2x_3^2 \geq 0$ and $-2x_1^2 + 5x_2^2 + 4x_3^2 \geq 0$ within the unit cube. Clearly, this case is more difficult to solve since this problem is non-convex and thus, we need to resort to the hierarchy of relaxations introduced in Theorem 5. Nevertheless, we have obtained good results in this case even with the first relaxation order.

4.1 Comparisons

In this section we report some partial comparative analysis of our algorithm with some others that have previously appeared in the literature. As we mentioned in the introduction, this comparison is not easy since most available algorithms for the ordered median problem are only applicable on the plane and with Euclidean norm (ℓ_2), with the only exception of [2] that also reports some results in $\mathbb{R}^3$. In addition, the different lambda parameters reported in those papers are not always the same or they are not applicable under the hypothesis of this paper. For instance, [24] reports only k-centrum and non-increasing lambda instances, [21] reports results for Weber, k-centrum and non-increasing lambda instances, whereas [2] does it for Weber, center and k-centrum, but does not report results for non-increasing lambda parameters.

Therefore, in order to have a meaningful comparative of methods we have selected different results from the above mentioned papers and we have organized them so that comparisons are as simple as possible.

In Table 6, we compare the results of the algorithms in Drezner and Nickel [21] (`DN09`), Espejo et al. [24], (`ERV09`), and the one in this paper (`New`) in terms of their CPU times on different instance sizes. The data that can be compared correspond to number of points ranging between n =100, 500, 1,000, and problem types Weber, k-centrum and non-increasing random lambda. These data appear in the three papers with the only exception of those for Weber problem that are not considered in [24] and therefore are marked as N/A in Table 6.

In addition, we also compare the results in this paper with those provided by the algorithm in [2], (`BHP12`), that is the only one that works for ordered median problems in dimension higher than 2. In this case, we report comparisons for $\tau = 2, 3$ and dimensions $d = 2, 3$ for problem types Weber, center and k-centrum. (All the data have been taken from those reported by the authors in the papers.)

Table 6 Comparison DN09, ERCV09 and New for Weber, k-centrum and random in dimension $d = 2$ and $\tau = 2$

n	Problem	Algorithm DN09	ERV09	New
100	Weber	0.47	N/A	0.05
	k-centrum	0.39	1.76	0.13
	Random	7.13	0.79	0.31
500	Weber	7.28	N/A	0.12
	k-centrum	3.99	5.63	0.68
	Random	85.56	4.69	12.88
1,000	Weber	27.69	N/A	0.42
	k-centrum	15.04	25.32	3.71
	Random	340.2	17.17	100.51

Table 7 Comparison between BHP09 and New for Weber, center and k-centrum problems and $\tau = 2, 3$

		$\mathbb{R}^2$				$\mathbb{R}^3$			
		τ				τ			
		2		3		2		3	
n	Problem	BHP12	New	BHP12	New	BHP12	New	BHP12	New
100	Weber	3.55	0.05	5.21	1.77	4.79	0.07	7.32	3.60
	Center	30.83	0.05	34.07	0.28	48.51	0.05	57.85	0.48
	k-centrum	37.58	0.13	34.41	0.66	52.52	0.10	53.87	1.46
500	Weber	17.74	0.12	27.46	10.82	25.32	0.14	37.22	17.87
	Center	305.36	0.09	299.41	14.47	566.29	0.10	600.27	34.85
	k-centrum	285.02	0.68	291.8	41.43	452.85	0.48	449.46	72.35
1,000	Weber	39.82	0.42	58.32	21.73	56.86	0.48	84.06	33.99
	Center	736.25	0.11	864.93	28.93	1494.76	0.16	1606.89	119.96
	k-centrum	666.2	3.71	729.3	100.17	1149.9	2.13	1280.1	145.24

From Table 6, we observe that ERCV09 specifically designed to work in dimension 2 is faster than ours; but only for random lambda parameters and when the number of considered points increases. This could have been expected since ERV09 takes explicit advantage of the geometry of the plane building explicitly planar bisectors among points. The reader should observe that this techniques is not applicable in higher dimensions because there are no algorithms to compute bisectors in dimension $d > 2$. On the other hand, New ensures a very high accuracy whereas the other two algorithms are heuristic and the precision with respect to the actual optimal solution is difficult to ensure.

Analyzing Table 7, we see that the New algorithm is much faster than BHP12 which is the only one valid in higher dimensions. Again, this is not surprising since BHP12 is based on global optimization techniques solving series of SDP relaxations which makes it applicable to even non-convex cases at the price of being slower and limited in problem size.

5 Conclusions

We develop a unified tool for minimizing convex ordered median location problems in finite dimension and with general ℓ_τ-norms. We report computational results that show the powerfulness of this methodology to solve medium size continuous location problems.

This new approach solves a broad class of convex and non-convex continuous location problems that, up to date, were only partially solved in the specialized literature. We have tested this methodology with some medium size standard ordered median location problems in different dimensions and with different norms.

It is important to emphasize that one of the contributions of our approach is that the same algorithm is used to solve all this family of location problems. This is an interesting novelty as compared with previous approaches, of course at the price of loosing some speed in the computations compared with some tailored algorithms for specific problems. Obviously, our goal was not to compete with previous algorithms since most of them are either designed for specific problems or only applicable for planar problems. However, in all cases we obtained reasonable CPU times and accurate results. Furthermore, in many cases our running times for many problems could not be even compared with others since nobody had solved them before.

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